

# ON THE NON-BALANCED PROPERTY OF THE CATEGORY OF CROSSED MODULES IN GROUPS.

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ABSTRACT. An algebraic category  $\mathcal{C}$  is called balanced if the cotriple cohomology of any object of  $\mathcal{C}$  vanishes in positive dimensions on injective coefficient modules. Important examples of balanced and of non-balanced categories occur in the literature. In this paper we prove that the category of crossed modules in groups is non-balanced.

## INTRODUCTION

The (co)homology theory of algebraic objects was studied using cotriple resolutions since the time of Beck's work [2]. In the category of groups cotriple (co)homology recovers, up to a dimension shift, ordinary group (co)homology, and similar results hold for Lie algebras and associative algebras over a field [1].

An axiomatization of the Barr-Beck setting was given by Orzech [9] who defined 'categories of interest', which include the examples mentioned above.

The interpretation of the cohomology groups  $H^n(G, A)$  of a group  $G$  with coefficients in a  $G$ -module  $A$  was given independently by various authors (see MacLane's historical note [7]) in terms of equivalence classes of crossed  $n$ -fold extensions of  $G$  by  $A$ .

In [13] Vale proved that cotriple cohomology in a category of interest can be interpreted in terms of equivalence classes of crossed  $n$ -fold extensions provided that the cotriple cohomology vanishes in positive dimensions on injective coefficients modules. A category of interest with this property is called, in this context, a balanced category.

Examples of balanced categories include groups and Lie algebras. The main example of a non-balanced category is commutative algebras. The corresponding cotriple theory is in this case André-Quillen cohomology [12].

Crossed modules in groups were introduced by Whitehead [14] as algebraic models of connected 2-types. One of the first studies of purely algebraic aspects of crossed modules in groups was the work of Norrie [8] which investigated actions internally to the category of crossed modules. In [3] the authors studied crossed modules as a category tripleable over the category of sets, and they

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introduced a corresponding (co)homology theory with trivial coefficients. A more general class of local coefficients was considered in [10].

The purpose of this paper is to answer the question whether the algebraic category of crossed modules in groups is balanced. In Theorem 8 we prove that crossed modules is non-balanced.

This paper is organized as follows. In Section 1 we provide some background. In 1.1 we recall a criterion, due to Cegarra [4], which is satisfied in any balanced category of interest. In 1.2 we recall how the category of crossed modules can be considered as a category of interest via its equivalence with the category of  $\text{cat}^1$ -groups.

In the first part of Section 2 we characterize modules in the category of crossed modules in groups, and reconcile the notion of action in the sense of categories of interest with the one described by Norrie. In the second part of Section 2 we investigate Cegarra's criterion in the category of crossed modules, and eventually prove that it does not hold, showing that crossed modules is non-balanced.

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## 1. PRELIMINARIES

**1.1. Categories of interest and Cegarra's criterion.** Let  $\mathcal{C}$  be a category of interest in the sense of Orzech [9]. Recall that this consists of a category satisfying the following axioms:

- 1) There is a triple  $(\top, \eta, \mu)$  on  $\mathbf{Set}$  such that  $\top(\emptyset) = \{\cdot\}$  (a one point set) and  $\mathcal{C}$  is equivalent to  $\mathbf{Set}^\top$ .

Let  $\mathbf{Set}_*$  denote the category of pointed sets, with basepoint preserving maps. From above,  $\mathcal{C}$  is pointed with  $(\top(\emptyset), \mu_\emptyset)$  as zero object. Also,  $\mathcal{C}$  is tripleable over  $\mathbf{Set}_*$ .

- 2) The underlying set functor  $U : \mathcal{C} \rightarrow \mathbf{Set}_*$  factors through the category of groups.
- 3) All operations in  $\mathcal{C}$  are finitary.

- 4) There is a generating set  $\Omega$  for the operations in  $\mathcal{C}$  and  $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ , where  $\Omega_i$  = set of  $i$ -ary operations in  $\Omega$ . Moreover  $\Omega$  includes the identity, inverse and multiplication associated with the group structure; let  $\Omega'_2 = \Omega_2 \setminus \{+\}$ ,  $\Omega'_1 = \Omega_1 \setminus \{-\}$ . We assume that if  $*$   $\in$   $\Omega'_2$  then  $*^o$  defined by  $x *^o y = y * x$  is also in  $\Omega'_2$ .
- 5) For each  $*$   $\in$   $\Omega'_2$ ,  $a * (b + c) = a * b + a * c$ .
- 6) For  $\omega \in \Omega'_1$ ,  $*$   $\in$   $\Omega'_2$ ,  $\omega(a + b) = \omega(a) + \omega(b)$  and  $\omega(a * b) = \omega(a) * b$ .
- 7) For each  $*$   $\in$   $\Omega'_2$ ,  $a + (b * c) = (b * c) + a$ .
- 8) For each ordered pair  $(\cdot, *) \in \Omega'_2 \times \Omega'_2$  there is a word  $w$  such that

$$(x_1 \cdot x_2) * x_3 = w(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, \\ x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where juxtaposition represents an operation in  $\Omega'_2$ .

We recall the notion of module in a category of interest. An object  $A$  of  $\mathcal{C}$  is called singular if  $A$  is an abelian group and if  $a_1 * a_2 = 0$  for all  $a_1, a_2 \in A$ ,  $*$   $\in$   $\Omega_2 \setminus \{+\}$ . Given an object  $R$  of  $\mathcal{C}$ , an  $R$ -module consists of a singular object  $A$  and of a split short exact sequence in  $\mathcal{C}$

$$A \hookrightarrow E \rightrightarrows R.$$

It follows that  $E \cong A \widetilde{\times} R$ , where  $A \widetilde{\times} R = A \times R$  as a set, with operations

$$\begin{aligned} \omega(a, r) &= (\omega(a), \omega(r)), \quad \text{for each } \omega \in \Omega_1 \setminus \{-\}, \\ (a', r') + (a, r) &= (a' + s(r') + a - s(r'), r' + r), \\ (a', r') * (a, r) &= (a' * a + a' * s(r) + s(r') * a, r' * r), \quad \text{for each } * \in \Omega_2 \setminus \{+\}. \end{aligned}$$

A *morphism of  $R$ -modules* consists of a morphism  $f : A \rightarrow A'$  in  $\mathcal{C}$  inducing a commutative diagram of split extensions

$$\begin{array}{ccccc} A & \longrightarrow & A \widetilde{\times} R & \longrightarrow & R \\ f \downarrow & & \downarrow (f, \text{id}) & & \parallel \\ A' & \longrightarrow & A' \widetilde{\times} R' & \longrightarrow & R \end{array}$$

The category  $R\text{-Mod}$  of  $R$ -modules in  $\mathcal{C}$  is equivalent to the category  $(\mathcal{C}/R)_{ab}$  of abelian group objects in  $\mathcal{C}/R$ . Further, it can be proved [4, p.30] that  $R\text{-Mod}$  has enough injectives.

Given an object  $Y \xrightarrow{f} R$  in the slice category  $\mathcal{C}/R$ , any  $R$ -module  $A$  is also a  $Y$ -module via the map  $f$ . There is a derivation functor  $\text{Der}(-, A) : \mathcal{C}/R \rightarrow \mathbf{Ab}$ , such that

$$\text{Der}(Y, A) \cong \text{Hom}_{\mathcal{C}/R}(Y, A \widetilde{\times} R). \quad (1)$$

An explicit description of a derivation  $D : Y \rightarrow A$  in  $\mathcal{C}$  can be given, see for instance [4, p.36]. The forgetful functor  $(\mathcal{C}/R)_{ab} \rightarrow \mathcal{C}/R$  has a left adjoint  $D_R : \mathcal{C}/R \rightarrow (\mathcal{C}/R)_{ab}$ . Since  $(\mathcal{C}/R)_{ab}$  is equivalent to  $R\text{-Mod}$ , and  $A \widetilde{\times} R \rightarrow R$

corresponds to  $A$  under this equivalence, by adjointness and by (1) it follows that

$$\mathrm{Der}(Y, A) \cong \mathrm{Hom}_{R\text{-Mod}}(D_R(Y), A). \quad (2)$$

Let  $\mathbb{G}$  be the cotriple in  $\mathcal{C}$  arising from the forgetful functor  $\mathcal{C} \rightarrow \mathbf{Set}$  and its left adjoint. This cotriple induces a cotriple on the slice category  $\mathcal{C}/R$  which we still denote by  $\mathbb{G}$ . The  $n^{\mathrm{th}}$  cotriple cohomology of  $R$  with coefficients in the  $R$ -module  $A$  is defined by

$$D^n(R, A) = H^n \mathrm{Der}(\mathbb{G}_* R, A)$$

where  $\mathbb{G}_* R \rightarrow R$  is the cotriple resolution. A category of interest  $\mathcal{C}$  is said to be *balanced* if for every object  $R$  of  $\mathcal{C}$ ,  $D^n(R, I) = 0$  for each  $n > 0$ , whenever  $I$  is an injective  $R$ -module.

Let  $N \hookrightarrow Y \xrightarrow{f} R$  be a short exact sequence in  $\mathcal{C}$ . The commutator subobject  $[N, N]$  is the ideal of  $N$  generated by the elements  $\{n_1 + n_2 - n_1 - n_2, n_1 * n_2 \mid n_1, n_2 \in N, * \in \Omega_2 \setminus \{+\}\}$ ; the quotient  $N/[N, N]$  is an  $R$ -module, with the action of  $R$  on  $N/[N, N]$  given by

$$r \cdot [n] = [y + n - y], \quad r * [n] = [r * n] \quad (3)$$

where  $f(y) = r$ ,  $y \in Y$ ,  $r \in R$ ,  $n \in N$ .

**Theorem 1.** [4, (1.7.11)] *Let  $\mathcal{C}$  be a category of interest,  $N \hookrightarrow Y \twoheadrightarrow R$  a short exact sequence in  $\mathcal{C}$ . Then there exists a natural exact sequence of  $R$ -modules*

$$\frac{N}{[N, N]} \xrightarrow{j} D_R(Y) \twoheadrightarrow D_R(R).$$

*Moreover, if  $\mathcal{C}$  is balanced, then  $j$  is a monomorphism.*

### Examples:

- a) Let  $\mathcal{C}$  be the category of groups, and let  $N \hookrightarrow G \twoheadrightarrow G'$  be a short exact sequence in groups. The corresponding 3-term exact sequence is  $N_{ab} \hookrightarrow \mathbb{Z}G' \otimes_{\mathbb{Z}G} \mathfrak{I}_G \twoheadrightarrow \mathfrak{I}_{G'}$ . Given a  $G$ -module  $A$ ,  $D^n(G, A) \cong H^{n+1}(G, A)$  for  $n > 0$  [2], hence  $D^n(G, I) = 0$  for  $n > 0$  and  $I$  an injective  $G$ -module. The category of groups is a balanced category.
- b) Let  $\mathcal{C}$  be the category of commutative algebras. Given the short exact sequence of commutative algebras  $N \hookrightarrow R \twoheadrightarrow R'$ , the corresponding 3-term exact sequence is  $N/N^2 \xrightarrow{j} R \otimes_A \Omega_{R'} \twoheadrightarrow \Omega_{R'}$ , where  $\Omega_R$  is the module of Kähler differentials. The map  $j$  is in general not injective, hence commutative algebras is an example of a non-balanced category.

**1.2. Crossed modules as a category of interest.** Recall that the category **CM** of crossed modules in groups has objects the triples  $(T, G, \mu)$  where  $\mu : T \rightarrow G$  is a group homomorphism,  $G$  acts on  $T$  and for all  $t, t' \in T$ ,  $g \in G$ ,

$$\mu({}^g t) = g\mu(t)g^{-1}, \quad \mu(t)t' = tt't^{-1}.$$

A morphism of crossed modules is a pair of group homomorphisms  $(f, h) : (T, G, \mu) \rightarrow (T', G', \mu')$  such that  $\mu'f = h\mu$  and  $f({}^g t) = {}^{h(g)}f(t)$ ,  $t \in T$ ,  $g \in G$ . The category **CM** is equivalent to the category  $\mathcal{C}^1\mathcal{G}$  of  $\text{cat}^1$ -groups [5]. Objects of  $\mathcal{C}^1\mathcal{G}$  are triples  $(G, d_0, d_1)$  where  $d_0, d_1 : G \rightarrow G$  are group homomorphisms and

$$d_0d_1 = d_1, \quad d_1d_0 = d_0, \quad [\ker d_0, \ker d_1] = 1. \quad (4)$$

A morphism of  $\text{cat}^1$ -groups  $f : (G, d_0, d_1) \rightarrow (G', d'_0, d'_1)$  is a group homomorphism  $f : G \rightarrow G'$  such that  $fd_i = d'_i f$ ,  $i = 0, 1$ . Since  $\ker d_i = \{d_i(x)x^{-1} \mid x \in G\}$ ,  $i = 0, 1$  the identities (4) are equivalent to

$$d_0d_1 = d_1, \quad d_1d_0 = d_0, \quad d_0(x)x^{-1}d_1(y)y^{-1} = d_1(y)y^{-1}d_0(x)x^{-1}, \quad x, y \in G.$$

It follows that  $\mathcal{C}^1\mathcal{G}$  is a category of universal algebras. The generating set of operations is  $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ ,  $\Omega_0 = \{0\}$ ,  $\Omega_1 = \{-\} \cup \{d_0, d_1\}$ ,  $\Omega_2 = \{+\}$ , where  $0, -, +$  denote group identity, inverse and multiplication and the following identities hold; for all  $x, y \in G$ ,

$$\begin{aligned} d_0(x+y) &= d_0(x) + d_0(y), & d_1(x+y) &= d_1(x) + d_1(y), \\ d_0d_1(x) &= d_1(x), & d_1d_0(x) &= d_0(x), \\ d_0(x)x^{-1}d_1(y)y^{-1} &= d_1(y)y^{-1}d_0(x)x^{-1}. \end{aligned}$$

It follows that the forgetful functor  $U : \mathcal{C}^1\mathcal{G} \rightarrow \mathbf{Set}$  has a left adjoint [6]. An explicit and very useful description of the left adjoint to  $U$  was given in [3].

The above description of  $\mathcal{C}^1\mathcal{G}$  also makes this category into a ‘category of interest’ in the sense of Orzech [9]. Since **CM** is equivalent to  $\mathcal{C}^1\mathcal{G}$ , we can therefore consider **CM** itself as a category of interest.

## 2. THE NON-BALANCED PROPERTY OF CROSSED MODULES.

### 2.1. The notion of module in the category **CM**.

**Definition 2.** Let  $(T, G, \mu)$  and  $(A, B, \delta)$  be crossed modules, and let  $\mathcal{G}_{(T, G, \mu)}$ ,  $\mathcal{G}_{(A, B, \delta)}$  be the corresponding  $\text{cat}^1$ -groups. We say that  $(A, B, \delta)$  is a  $(T, G, \mu)$ -module if  $\mathcal{G}_{(A, B, \delta)}$  is a  $\mathcal{G}_{(T, G, \mu)}$ -module in the sense of categories of interest.

**Lemma 3.** Given crossed modules  $(T, G, \mu)$  and  $(A, B, \delta)$ , the following are equivalent:

- i)  $(A, B, \delta)$  is a  $(T, G, \mu)$ -module.

ii)  $(A, B, \delta)$  is an abelian crossed module and there is a split short exact sequence in **CM**

$$(A, B, \delta) \xrightarrow{i} (T', G', \mu') \xrightleftharpoons[s]{\quad} (T, G, \mu) \quad (5)$$

iii)  $(A, B, \delta)$  is an abelian crossed module and there is an action of  $(T, G, \mu)$  on  $(A, B, \delta)$  in the sense of [8], that is there is a crossed module morphism  $(\varepsilon, \rho) : (A, B, \delta) \rightarrow \text{Act}(T, G, \mu)$ .

**Proof.**

i)  $\Leftrightarrow$  ii)

By definition  $(A, B, \delta)$  is a  $(T, G, \mu)$ -module if and only if  $\mathcal{G}_{(A, B, \delta)}$  is a singular object in  $\mathcal{C}^1\mathcal{G}$  and there is a split short exact sequence in  $\mathcal{C}^1\mathcal{G}$  of the form

$$\mathcal{G}_{(A, B, \delta)} \hookrightarrow \mathcal{G} \xrightleftharpoons[\quad]{\quad} \mathcal{G}_{(T, G, \mu)}. \quad (6)$$

Here  $\mathcal{G}_{(A, B, \delta)} = (A \rtimes B, d_0, d_1)$ ,  $d_0(a, b) = (1, b)$ ,  $d_1(a, b) = (1, \delta(a)b)$ . By definition,  $\mathcal{G}_{(A, B, \delta)}$  is a singular object in  $\mathcal{C}^1\mathcal{G}$  if and only if  $A \rtimes B$  is an abelian group. This is equivalent to  $(A, B, \delta)$  being an abelian crossed module. In fact, if  $A \rtimes B$  is abelian, both  $A$  and  $B$  are abelian groups and for all  $a \in A$ ,  $b \in B$   $(a, b) = (a, 0) + (0, b) = (0, b) + (a, 0) = ({}^b a, b)$  so that  $a = {}^b a$ . Hence  $(A, B, \delta)$  is an abelian crossed module. Conversely, if  $(A, B, \delta)$  is an abelian crossed module, then  $A \rtimes B = A \oplus B$  is an abelian group.

Let  $(T', G', \mu')$  be the crossed module corresponding to the  $\text{cat}^1$ -group  $\mathcal{G}$ . Then (6) holds if and only if there exists a split short exact sequence in **CM** of the form

$$(A, B, \delta) \hookrightarrow (T', G', \mu') \xrightleftharpoons[\quad]{\quad} (T, G, \mu). \quad (7)$$

In conclusion i) is equivalent to the existence of a short exact sequence (6) with  $\mathcal{G}_{(A, B, \delta)}$  a singular  $\text{cat}^1$ -group; in turn this is equivalent to the short exact sequence (7) with  $(A, B, \delta)$  an abelian crossed module, which is ii).

ii)  $\Leftrightarrow$  iii)

If ii) holds, then the split short exact sequence (5) induces split short exact sequences of groups  $A \hookrightarrow T' \xrightleftharpoons[\quad]{\quad} T$  and  $B \hookrightarrow G' \xrightleftharpoons[\quad]{\quad} G$ , so that there are induced actions of  $T$  on  $A$  and of  $G$  on  $B$ , and  $T' = A \rtimes T$ ,  $G' \cong B \rtimes G$ . Since the maps  $i$  and  $s$  in (5) are injective, we can identify  $(T, G, \mu)$  with  $s(T, G, \mu)$  and  $(A, B, \delta)$  with  $i(A, B, \delta)$ . Hence, by (5), we can regard  $(A, B, \delta)$  and  $(T, G, \mu)$  as subcrossed modules of  $(T', G', \mu')$ . With these identifications, we have

- a)  $(A, B, \delta)$  is a normal subcrossed module of  $(T', G', \mu')$ .
- b)  $T' = AT$ ,  $G' = BG$ .
- c)  $A \cap T = 1$ ,  $B \cap G = 1$ .

By [8, p. 135] it follows that there is a morphism of crossed modules  $(\varepsilon, \rho) : (A, B, \delta) \rightarrow \text{Act}(T, G, \mu)$ ,  $\varepsilon_a(g) = a - {}^g a$ ,  $\rho(b) = (\alpha_b, \gamma_b)$ ,  $\alpha_b(t) = {}^b t$ ,  $\gamma_b(t) = btb^{-1}$ . Hence ii) implies iii).

Conversely, if iii) holds, by [8] there exists a split short exact sequence of crossed modules

$$(A, B, \delta) \hookrightarrow (A \rtimes T, B \rtimes G, (\delta, \mu)) \rightrightarrows (T, G, \mu)$$

where the crossed module action of  $B \rtimes G$  on  $A \rtimes T$  is given by

$$({}^{b,g})(a, t) = ({}^b({}^g a) - \varepsilon({}^g t)(b), {}^g t) = ({}^g a - \varepsilon({}^g t)(b), {}^g t).$$

Hence ii) holds.  $\square$

**Definition 4.** Let  $(A, B, \delta)$ ,  $(A', B', \delta')$  be  $(T, G, \mu)$ -modules. A morphism  $f : (A, B, \delta) \rightarrow (A', B', \delta')$  of  $(T, G, \mu)$ -modules is a crossed module morphism such that the corresponding morphism of cat<sup>1</sup>-groups  $f : \mathcal{G}_{(A,B,\delta)} \rightarrow \mathcal{G}_{(A',B',\delta')}$  is a morphism of  $\mathcal{G}_{(A,B,\delta)}$ -modules in the sense of categories of interest.

**Lemma 5.** A morphism of  $(T, G, \mu)$ -modules  $(r, s) : (A, B, \delta) \rightarrow (A', B', \delta')$  consists of a pair of morphisms of abelian groups such that

$$\delta' r = s \delta, \quad r({}^g a) - r(\varepsilon({}^g t)(b)) = {}^g r(a) - \varepsilon'({}^g t)(s(b))$$

for  $a \in A$ ,  $b \in B$ ,  $g \in G$ ,  $t \in T$ , where  $(\varepsilon, \rho) : (A, B, \delta) \rightarrow \text{Act}(T, G, \mu)$  and  $(\varepsilon', \rho') : (A', B', \delta') \rightarrow \text{Act}(T, G, \mu)$ .

**Proof.** By Definition 4 and Lemma 5,  $(r, s)$  is a morphism of  $(T, G, \mu)$ -modules if and only if there is a commutative diagram of split short exact sequences

$$\begin{array}{ccccc} (A, B, \delta) & \longrightarrow & (A \rtimes T, B \rtimes G, (\delta, \mu)) & \rightrightarrows & (T, G, \mu) \\ (r, s) \downarrow & & \downarrow ((r, \text{id}_T), (s, \text{id}_G)) & & \parallel \\ (A', B', \delta') & \longrightarrow & (A' \rtimes T, B' \rtimes G, (\delta', \mu')) & \rightrightarrows & (T, G, \mu) \end{array}$$

in **CM**. This is equivalent to  $(r, s)$  and  $((r, \text{id}_T), (s, \text{id}_G))$  being crossed module morphisms. Since  $(A, B, \delta)$  and  $(A', B', \delta')$  are abelian crossed modules,  $(r, s)$  is a crossed module morphism if and only if  $\delta' r = s \delta$ . For  $((r, \text{id}_T), (s, \text{id}_G))$  to be a crossed module morphism we further require that

$$(r, \text{id}_T)({}^{(b,g)}(a, t)) = {}^{(s(b),g)}(r(a), t)$$

for all  $a \in A$ ,  $b \in B$ ,  $t \in T$ ,  $g \in G$ . Hence

$$(r, \text{id}_T)({}^g a - \varepsilon({}^g t)(b), {}^g t) = ({}^g r(a) - \varepsilon'({}^g t)(s(b)), {}^g t);$$

that is

$$r({}^g a) - r(\varepsilon({}^g t)(b)) = {}^g r(a) - \varepsilon'({}^g t)(s(b)).$$

$\square$

## 2.2. Crossed modules is non-balanced.

**Lemma 6.** *Let  $M$  be an abelian group and consider an extension of crossed modules*

$$(N, G, \nu) \hookrightarrow (T, G, \mu) \xrightarrow{(f,0)} (M, 1, 0).$$

*Let  $J = [G, N][T, T]$ . Then  $(T/J, G_{ab}, \bar{\mu})$  is a  $(M, 1, 0)$ -module with action  $(\varepsilon', 1) : (M, 1, 0) \rightarrow \text{Act}(T/J, G_{ab}, \bar{\mu})$ ,  $\bar{\mu}[t] = [\mu(t)]$ ,  $\varepsilon'(m)[g] = [t] - [{}^g t]$ ,  $m = f(t)$ ,  $t \in T$ ,  $g \in G$ . Moreover,*

$$\text{Der}((T, G, \mu), (A, B, \delta)) \cong \text{Hom}_{(M,1,0)\text{-Mod}}((T/J, G_{ab}, \bar{\mu}), (A, B, \delta)). \quad (8)$$

**Proof.** We first check that

$$(\varepsilon', 1) : (M, 1, 0) \rightarrow \text{Act}(T/J, G_{ab}, \bar{\mu})$$

is a crossed module morphism. We recall from [8] that

$$\text{Act}(T/J, G_{ab}, \bar{\mu}) = (D(G_{ab}, T/J), \text{Aut}(T/J, G_{ab}, \bar{\mu})(\theta, \sigma)),$$

where  $D(G_{ab}, T/J)$  is the group of Whitehead derivations.

The map  $\varepsilon'$  is well defined. In fact, if  $m = f(t_1) = f(t_2)$ , then  $t_1 t_2^{-1} \in N$  so that  $[{}^g(t_1 t_2^{-1})] = [t_1 t_2^{-1}]$ ,  $t_1 t_2 \in T$ ,  $g \in G$ . Hence

$$\varepsilon'(f(t_1))[g] = [t_1] - [{}^g t_1] = [t_2] - [{}^g t_2] = \varepsilon'(f(t_2))[g]. \quad (9)$$

Choosing  $t_1 = t$ ,  $t_2 = {}^{g^{-1}}t$  in (9), we obtain  $[{}^g t] - [t] + [{}^{g^{-1}}t] - [t] = 0$  for all  $t \in T$ ,  $g \in G$ . Hence, for all  $g_1, g_2 \in G$ ,  $t \in T$

$$\begin{aligned} [{}^{g_1 g_2 g_1^{-1} g_2^{-1}} t] - [t] &= [{}^{g_1 g_2 g_1^{-1} g_2^{-1}} t] - [{}^{g_2 g_1^{-1} g_2^{-1}} t] + \\ &+ [{}^{g_2 g_1^{-1} g_2^{-1}} t] - [{}^{g_1^{-1} g_2^{-1}} t] + [{}^{g_1^{-1} g_2^{-1}} t] - [{}^{g_2^{-1}} t] + [{}^{g_2^{-1}} t] - [t] = \\ &= [{}^{g_1} t] - [t] + [{}^{g_2} t] - [t] + [{}^{g_1^{-1}} t] - [t] + [{}^{g_2^{-1}} t] - [t] = 0. \end{aligned}$$

This proves that  $[{}^x t] = [t]$  for all  $x \in [G, G]$ ,  $t \in T$ . Hence, if  $[g_1] = [g_2]$ ,  $\varepsilon'(m)[g_1] = \varepsilon'(m)[g_2]$ .

Notice that  $\varepsilon'(m) \in D(G_{ab}, T/J)$ . In fact,  $\text{Im } \varepsilon'(m) \subseteq \ker \bar{\mu}$ , since  $\bar{\mu}([t] - [{}^g t]) = [\mu(t)] - [g\mu(t)g^{-1}] = 0$ . It follows easily from [8] that  $\varepsilon'(m)$  is a unit in  $\text{Der}(G_{ab}, T/J)$ , hence is in  $D(G_{ab}, T/J)$ . By [8] for each  $t, t' \in T$ ,  $g \in G$ ,  $m = f(t)$ ,

$$\begin{aligned} \theta \varepsilon'(m)[t'] &= \varepsilon'(f(t))\bar{\mu}[t'] + [t'] = [t] - [{}^{\mu(t')} t] + [t'] = [t] - [t' t t'^{-1}] + [t'] = [t'], \\ \sigma \varepsilon'(f(t))[g] &= \bar{\mu} \varepsilon'(f(t))[g] + [g] = \bar{\mu}([t] - [{}^g t]) + [g] = [g]. \end{aligned}$$

Hence  $(\theta, \sigma)\varepsilon' = \text{id}$ , so that  $(\varepsilon', 1)$  is a crossed module morphism.

By Lemma 3, a  $(M, 1, 0)$ -module consists of an abelian crossed module  $(A, B, \delta)$  and of a crossed module map  $(\varepsilon, 1) : (M, 1, 0) \rightarrow \text{Act}(A, B, \delta) = (D(B, A), \text{Aut}(A, B, \delta), (\vartheta, \sigma))$ . The corresponding split extension in **CM** of  $(M, 1, 0)$  by  $(A, B, \delta)$  has the form

$$(A, B, \delta) \hookrightarrow (A \oplus M, B, \bar{\delta}) \rightrightarrows (M, 1, 0)$$

$\bar{\delta}(a, m) = \delta(a)$ ,  ${}^b(a, m) = (a - \varepsilon(m)(b), m)$ ,  $a \in A$ ,  $b \in B$ ,  $m \in M$ .

By Lemma 5,  $(r, s) : (A, B, \delta) \rightarrow (A', B', \delta')$  is a morphism of  $(M, 1, 0)$ -modules if and only if  $r, s$  are homomorphisms of abelian groups and

$$\delta' r = s \delta, \quad r(\varepsilon(m)(b)) = \varepsilon'(m)(s(b)), \quad b \in B, m \in M, \quad (10)$$

where  $\varepsilon$  and  $\varepsilon'$  are as in Lemma 5. By (1),

$$\text{Der}((T, G, \mu), (A, B, \delta)) \cong \text{Hom}_{\mathbf{CM}/(M, 1, 0)}((T, G, \mu), (A \oplus M, B, \bar{\delta})). \quad (11)$$

We aim to show that

$$\begin{aligned} \text{Hom}_{\mathbf{CM}/(M, 1, 0)}((T, G, \mu), (A \oplus M, B, \bar{\delta})) &\cong \\ &\cong \text{Hom}_{(M, 1, 0)\text{-Mod}}((T/J, G_{ab}, \bar{\mu}), (A, B, \delta)). \end{aligned} \quad (12)$$

Let take  $((D_1, f), D_2) \in \text{Hom}_{\mathbf{CM}/(M, 1, 0)}((T, G, \mu), (A \oplus M, B, \bar{\delta}))$ . Then  $D_1 \in \text{Der}(T, A)$ ,  $D_2 \in \text{Der}(G, B)$  and for all  $t \in T$ ,  $g \in G$

$$D_2 \mu = \delta D_1, \quad D_1({}^g t) = D_1(t) - \varepsilon(f(t))(D_2(g)). \quad (13)$$

Let  $\varphi((D_1, f), D_2) = (\nu_1, \nu_2)$  where  $\nu_1 \in \text{Hom}_{\mathbb{Z}}(T/J, A)$ ,  $\nu_2 \in \text{Hom}_{\mathbb{Z}}(G_{ab}, B)$  are defined by  $\nu_1[t] = D_1(t)$ ,  $\nu_2[g] = D_2(g)$ ,  $t \in T$ ,  $g \in G$ .

Since  $(T, G, \mu)$  acts on  $(A, B, \delta)$  via  $(f, 0)$  and  $M$  acts trivially on  $A$ , then  $T$  acts trivially on  $A$  and  $G$  acts trivially on  $B$ . Hence  $D_1(x) = 0$  for all  $x \in [T, T]$  and  $D_2(y) = 0$  for all  $y \in [G, G]$ . Moreover from (13)  $D_1({}^g n) = D_1(n)$  for all  $n \in N$ ,  $g \in G$ , so that  $D_1(x) = 0$  for all  $x \in J$ . It follows easily that  $\nu_1, \nu_2$  are well defined. By (13),

$$\begin{aligned} \nu_2 \bar{\mu}[t] &= \nu_2[\mu(t)] = D_2 \mu(t) = \delta D_1(t) = \delta \nu_1[t] \\ \nu_1[{}^g t] &= \nu_1[t] - \varepsilon(f(t))(\nu_2[g]). \end{aligned}$$

From the definition of  $\varepsilon'$ , we conclude that

$$\nu_2 \bar{\mu} = \delta \nu_1, \quad \nu_1(\varepsilon'(m))[g] = \varepsilon(m)(\nu_2[g]), \quad m = f(t). \quad (14)$$

Hence (10) holds and  $(\nu_1, \nu_2)$  is a morphism of  $(M, 1, 0)$ -modules.

Conversely, if  $(\nu_1, \nu_2) \in \text{Hom}_{(M, 1, 0)\text{-Mod}}((T/J, G_{ab}, \bar{\mu}), (A, B, \delta))$  then (14) holds. Let  $\psi(\nu_1, \nu_2) = ((D_1, f), D_2)$  where  $D_1(t) = \nu_1[t]$ ,  $D_2(g) = \nu_2[g]$ . Since  $T$  acts trivially on  $A$  and  $B$  acts trivially on  $G$ ,  $D_1 \in \text{Der}(T, A)$  and  $D_2 \in \text{Der}(G, B)$ . From (14),  $D_2 \mu = \delta D_1$  while taking  $m = f(t)$  and using the definition of  $\varepsilon'$  we obtain from the second identity in (14)

$$D_1({}^g t) = D_1(t) - \varepsilon(f(t))(D_2(g)).$$

Hence  $((D_1, f), D_2) \in \text{Hom}_{\mathbf{CM}/(M, 1, 0)}((T, G, \mu), (A \oplus M, B, \bar{\delta}))$ .

It is straightforward to check that  $\psi\varphi = \text{id}$  and  $\varphi\psi = \text{id}$ , proving the isomorphism (12). By (11), (8) follows.  $\square$

By the previous lemma, the left adjoint to the forgetful functor  $(\mathbf{CM}/(M, 1, 0))_{ab} \rightarrow \mathbf{CM}/(M, 1, 0)$  is given by

$$D_{(M,1,0)}(T, G, \mu) = (T/J, G_{ab}, \bar{\mu})$$

where  $(T, G, \mu) \rightarrow (M, 1, 0)$  is an object of  $\mathbf{CM}/(M, 1, 0)$  and  $J$  is as in Lemma (6).

**Proposition 7.** *Given the short exact sequence of crossed modules*

$$(N, G, \nu) \hookrightarrow (T, G, \mu) \xrightarrow{(f,0)} (M, 1, 0)$$

*there is an exact sequence of  $(M, 1, 0)$ -modules*

$$(N/[G, N], G_{ab}, \bar{\nu}) \xrightarrow{(u, \text{id})} (T/J, G_{ab}, \bar{\mu}) \xrightarrow{(\bar{f}, 0)} (M, 1, 0)$$

where  $J = [G, N][T, T]$ ,  $u(n[G, N]) = [n]$ ,  $\bar{f}([t]) = f(t)$ ,  $n \in N$ ,  $t \in T$ . The  $(M, 1, 0)$ -module structure of  $(T/J, G_{ab}, \bar{\mu})$  is as in Lemma 6, and the  $(M, 1, 0)$ -module structure of  $(N/[G, N], G_{ab}, \bar{\nu})$  is given by  $(\varepsilon'', 1) : (M, 1, 0) \rightarrow \text{Act}(T/J, G_{ab}, \bar{\mu})$ , where  $\varepsilon''(m)[g] = [t^g t^{-1}]$ ,  $m = f(t)$ ,  $g \in G$ ,  $t \in T$ .

**Proof.** Let  $\mathcal{G}_{(N,G,\nu)}$ ,  $\mathcal{G}_{(T,G,\mu)}$ ,  $\mathcal{G}_{(M,1,0)}$  be the  $\text{cat}^1$ -groups corresponding to the crossed modules  $(N, G, \nu)$ ,  $(T, G, \mu)$ ,  $(M, 1, 0)$  respectively. By Theorem 1, there is an exact sequence of  $\mathcal{G}_{(M,1,0)}$ -modules

$$\mathcal{G}_{(N,G,\nu)} / [\mathcal{G}_{(N,G,\nu)}, \mathcal{G}_{(N,G,\nu)}] \rightarrow D_{\mathcal{G}_{(M,1,0)}} \mathcal{G}_{(T,G,\mu)} \twoheadrightarrow D_{\mathcal{G}_{(M,1,0)}} \mathcal{G}_{(M,1,0)}.$$

In the equivalent category of  $(M, 1, 0)$ -modules,  $D_{\mathcal{G}_{(M,1,0)}} \mathcal{G}_{(T,G,\mu)}$  corresponds to  $D_{(M,1,0)}(T, G, \mu) = (T/J, G_{ab}, \bar{\mu})$  and  $D_{\mathcal{G}_{(M,1,0)}} \mathcal{G}_{(M,1,0)}$  corresponds to  $D_{(M,1,0)}(M, 1, 0) = (M, 1, 0)$ .

Since  $\mathcal{G}_{(N,G,\nu)} = (N \rtimes G, s_0, s_1)$ ,  $s_0(n, g) = (1, g)$ ,  $s_1(n, g) = (1, \nu(n)g)$ , it is

$$\mathcal{G}_{(N,G,\nu)} / [\mathcal{G}_{(N,G,\nu)}, \mathcal{G}_{(N,G,\nu)}] \cong ((N \rtimes G)_{ab}, \bar{s}_0, \bar{s}_1).$$

where  $\bar{s}_i$  is induced by  $s_i$ ,  $i = 0, 1$ . On the other hand it is not hard to check that the map

$$\varphi : \left( \frac{N}{[G, N]} \rtimes \frac{G}{[G, G]}, u_0, u_1 \right) \rightarrow ((N \rtimes G)_{ab}, \bar{s}_0, \bar{s}_1) \quad (15)$$

$\varphi([n], [g]) = [(n, g)]$ ,  $u_0([n], [g]) = (0, [g])$ ,  $u_1([n], [g]) = (0, [\nu(n)g])$ , is well defined and it is an isomorphism in  $\mathcal{C}^1\mathcal{G}$ . Hence the crossed module corresponding to  $\mathcal{G}_{(N,G,\nu)} / [\mathcal{G}_{(N,G,\nu)}, \mathcal{G}_{(N,G,\nu)}]$  is  $(N/[G, N], G_{ab}, \bar{\nu})$ .

By (3) the  $\mathcal{G}_{(M,1,0)}$ -module structure of  $\mathcal{G}_{(N,G,\nu)} / [\mathcal{G}_{(N,G,\nu)}, \mathcal{G}_{(N,G,\nu)}]$  is given by

$$m \cdot [(n, g)] = [(t, 1)(n, g)(t, 1)^{-1}] = [(tn^g t^{-1}, g)]$$

where  $f(t) = m$ ,  $[(n, g)] \in (N \rtimes G)_{ab}$ ,  $t \in T$ .

From the isomorphism (15), the action of  $\mathcal{G}_{(M,1,0)}$  on  $(\frac{N}{[G, N]} \rtimes \frac{G}{[G, G]}, u_0, u_1)$  is therefore

$$m \cdot ([n], [g]) = ([tn^g t^{-1}], [g]) \quad (16)$$

$m = f(t)$ ,  $t \in T$ ,  $g \in G$ ,  $n \in N$ . Consider the semidirect product in the category of interest  $\mathcal{C}^1\mathcal{G}$ :

$$\frac{\mathcal{G}_{(N,G,\nu)}}{[\mathcal{G}_{(N,G,\nu)}, \mathcal{G}_{(N,G,\nu)}]} \widetilde{\times} \mathcal{G}_{(M,1,0)} \cong \left( \left( \frac{N}{[G, N]} \rtimes G_{ab} \right) \rtimes M, (u_0, 0), (u_1, 0) \right).$$

The crossed module corresponding to this semidirect product is

$$(\ker(u_0, 0), \text{Im}(u_1, 0), (u_1, 0)|_{\ker(u_0, 0)})$$

with the action by conjugation of  $\text{Im}(u_1, 0)$  on  $\ker(u_0, 0)$ . Hence we obtain from (16) the crossed module action

$$\begin{aligned} ((1, [g]), 0)(([n], 1), m) &= ((1, [g]), 0)(([n], 1), m)((1, [g^{-1}]), 0) = \\ &= (([n], [g]), m)((1, [g^{-1}]), 0) = (([n], [g])([t^{g^{-1}}t^{-1}], [g^{-1}]), m) = \\ &= ([n] + [{}^g t t^{-1}], 1), m). \end{aligned}$$

There is clearly an isomorphism

$$(\ker(u_0, 0), \text{Im}(u_1, 0), (u_1, 0)|_{\ker(u_0, 0)}) \cong \left( \frac{N}{[G, N]} \rtimes M, G_{ab}, (\bar{\nu}, 0) \right).$$

Hence, by the previous calculation, the crossed module action of  $G_{ab}$  on  $\frac{N}{[G, N]} \rtimes M$  is given by

$$[g]([n], m) = ([n] + [{}^g t t^{-1}], m)$$

$g \in G$ ,  $n \in N$ ,  $m = f(t)$ ,  $t \in T$ . On the other hand, by [8], the crossed module action in the semidirect product crossed module  $(\frac{N}{[G, N]}, G_{ab}, \nu) \rtimes (M, 1, 0) \cong (\frac{N}{[G, N]} \rtimes M, G_{ab}, (\bar{\nu}, 0))$  is given by

$$[g]([n], m) = ([n] - \varepsilon''(m)[g], m)$$

where  $(\varepsilon'', 1) : (M, 1, 0) \rightarrow \text{Act}(T/J, G_{ab}, \bar{\mu})$ . We deduce  $\varepsilon''(m)[g] = -[{}^g t t^{-1}] = [t {}^g t^{-1}]$ ,  $m = f(t)$ ,  $g \in G$ ,  $t \in T$ .  $\square$

**Theorem 8.** *The category of crossed modules in groups is non-balanced.*

**Proof.** We are going to show that there exists an extension of crossed modules such that the corresponding 3-terms exact sequence is not short exact. By Theorem 1 this proves that crossed modules is non-balanced.

Let  $M$  be an abelian group with  $H_2(M) \neq 0$ , and let  $f : T \rightarrow M$  be a surjection with  $T$  a free group. Let  $N = \ker f$ . Consider the extension of crossed modules

$$(N, T, i) \hookrightarrow (T, T, \text{id}) \xrightarrow{(f, 0)} (M, 1, 0) \quad (17)$$

where  $i$  denotes the inclusion. The well known exact sequence  $H_2(T) \rightarrow H_2(M) \rightarrow N/[T, N] \rightarrow T_{ab} \twoheadrightarrow M_{ab}$  reduces to

$$H_2(M) \hookrightarrow N/[T, N] \xrightarrow{j} T_{ab} \twoheadrightarrow M$$

where  $j(n[T, N]) = n[T, T]$ ,  $n \in N$ . Since  $H_2(M) \neq 0$ , the map  $j$  is not injective. By Proposition 7, the 3-terms exact sequence associated to the extension (17) is

$$(N/[T, N], T_{ab}, i) \xrightarrow{(j, \text{id})} (T_{ab}, T_{ab}, \text{id}) \xrightarrow{(\bar{f}, 0)} (M, 1, 0)$$

and is not short exact. □

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